

# Structure and applications of real $C^*$ -algebras

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*Dedicated to Dick Kadison, with admiration and appreciation*

ABSTRACT. For a long time, practitioners of the art of operator algebras always worked over the complex numbers, and nobody paid much attention to real  $C^*$ -algebras. Over the last thirty years, that situation has changed, and it's become apparent that real  $C^*$ -algebras have a lot of extra structure not evident from their complexifications. At the same time, interest in real  $C^*$ -algebras has been driven by a number of compelling applications, for example in the classification of *manifolds of positive scalar curvature*, in *representation theory*, and in the study of *orientifold string theories*. We will discuss a number of interesting examples of these, and how the real Baum-Connes conjecture plays an important role.

## 1. Real $C^*$ -algebras

DEFINITION 1.1. A *real  $C^*$ -algebra* is a Banach  $*$ -algebra  $A$  over  $\mathbb{R}$  isometrically  $*$ -isomorphic to a norm-closed  $*$ -algebra of bounded operators on a real Hilbert space.

REMARK 1.2. There is an equivalent abstract definition: a real  $C^*$ -algebra is a real Banach  $*$ -algebra  $A$  satisfying the  $C^*$ -identity  $\|a^*a\| =$

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$\|a\|^2$  (for all  $a \in A$ ) and *also* having the property that for all  $a \in A$ ,  $a^*a$  has spectrum contained in  $[0, \infty)$ , or equivalently, having the property that  $\|a^*a\| \leq \|a^*a + b^*b\|$  for all  $a, b \in A$  [32, 48].

Books dealing with real  $C^*$ -algebras include [25, 63, 39], though they all have a slightly different emphasis from the one presented here.

**THEOREM 1.3** (“Schur’s Lemma”). *Let  $\pi$  be an irreducible representation of a real  $C^*$ -algebra  $A$  on a real Hilbert space  $\mathcal{H}$ . Then the commutant  $\pi(A)'$  of the representation must be  $\mathbb{R}$ ,  $\mathbb{C}$ , or  $\mathbb{H}$ .*

**PROOF.** Since  $\pi(A)'$  is itself a real  $C^*$ -algebra (in fact a real von Neumann algebra), it is enough to show it is a division algebra over  $\mathbb{R}$ , since by Mazur’s Theorem [42] (variants of the proof are given in [32, Theorem 3.6] and [9]),  $\mathbb{R}$ ,  $\mathbb{C}$ , and  $\mathbb{H}$  are the only normed division algebras over  $\mathbb{R}$ .<sup>1</sup> Let  $x$  be a self-adjoint element of  $\pi(A)'$ . If  $p \in \pi(A)'$  is a spectral projection of  $x$ , then  $p\mathcal{H}$  and  $(1-p)\mathcal{H}$  are both invariant subspaces of  $\pi(A)$ . By irreducibility, either  $p = 1$  or  $1-p = 1$ . So this shows  $x$  must be of the form  $\lambda \cdot 1$  with  $\lambda \in \mathbb{R}$ . Now if  $y \in \pi(A)'$ ,  $y^*y = \lambda \cdot 1$  with  $\lambda \in \mathbb{R}$ , and similarly,  $yy^*$  is a real multiple of 1. Since the spectra of  $y^*y$  and  $yy^*$  must coincide except perhaps for 0,  $y^*y = yy^* = \lambda = \|y\|^2$  and either  $y = 0$  or else  $y$  is invertible (with inverse  $\|y\|^{-2}y^*$ ). So  $\pi(A)'$  is a division algebra.  $\square$

**COROLLARY 1.4.** *The irreducible  $*$ -representations of a real  $C^*$ -algebra can be classified into three types: real, complex, and quaternionic. (All of these can occur, as one can see from the examples of  $\mathbb{R}$ ,  $\mathbb{C}$ , and  $\mathbb{H}$  acting on themselves by left translation.)*

Given a real  $C^*$ -algebra  $A$ , its complexification  $A_{\mathbb{C}} = A + iA$  is a complex  $C^*$ -algebra, and comes with a real-linear  $*$ -automorphism  $\sigma$  with  $\sigma^2 = 1$ , namely complex conjugation (with  $A$  as fixed points). Alternatively, we can consider  $\theta(a) = \sigma(a^*) = (\sigma(a))^*$ . Then  $\theta$  is a (complex linear)  $*$ -antiautomorphism of  $A_{\mathbb{C}}$  with  $\theta^2 = 1$ . Thus we can classify real  $C^*$ -algebras by classifying their complexifications and then classifying all possibilities for  $\sigma$  or  $\theta$ . This raises a number of questions:

**PROBLEM 1.5.** Given a complex  $C^*$ -algebra  $A$ , is it the complexification of a real  $C^*$ -algebra? Equivalently, does it admit a  $*$ -antiautomorphism  $\theta$  with  $\theta^2 = 1$ ?

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<sup>1</sup>Historical note: According to [37], Mazur presented this theorem in Lwów in 1938. Because of space limitations in *Comptes Rendus*, he never published the proof, but his original proof is reproduced in [73] as well as in [41], which also includes a copy of Mazur’s original hand-typed manuscript, with the proof included.

The answer to this in general is no. For example, Connes [11, 12] showed that there are factors not anti-isomorphic to themselves, hence admitting no real form. Around the same time (ca. 1975), Philip Green (unpublished) observed that a stable continuous-trace algebra over  $X$  with Dixmier-Douady invariant  $\delta \in H^3(X, \mathbb{Z})$  cannot be anti-isomorphic to itself unless there is a self-homeomorphism of  $X$  sending  $\delta$  to  $-\delta$ . Since it is easy to arrange for this not to be the case, there are continuous-trace algebras not admitting a real form.

By the way, just because a factor is anti-isomorphic to itself, that does not mean it has an self-antiautomorphism of period 2, and so it may not admit a real form. Jones constructed an example in [33].

Secondly we have:

**PROBLEM 1.6.** Given a complex  $C^*$ -algebra  $A$  that admits a real form, how many distinct such forms are there? Equivalently, how many conjugacy classes are there of  $*$ -antiautomorphisms  $\theta$  with  $\theta^2 = 1$ ?

In general there can be more than one class of real forms. For example,  $M_2(\mathbb{C})$  is the complexification of two distinct real  $C^*$ -algebras,  $M_2(\mathbb{R})$  and  $\mathbb{H}$ . From this one can easily see that  $\mathcal{K}(\mathcal{H})$  and  $\mathcal{B}(\mathcal{H})$ , the compact and bounded operators on a separable infinite-dimensional Hilbert space, each have two distinct real forms. For example,  $\mathcal{K}(\mathcal{H})$  is the complexification of both  $\mathcal{K}(\mathcal{H}_{\mathbb{R}})$  and  $\mathcal{K}(\mathcal{H}_{\mathbb{H}})$ . This makes the following theorem due independently to Størmer and to Giordano and Jones all the more surprising and remarkable.

**THEOREM 1.7** ([67, 22, 20]). *The hyperfinite  $II_1$  factor  $R$  has a unique real form.*

This has a rather surprising consequence: if  $R_{\mathbb{R}}$  is the (unique) real hyperfinite  $II_1$  factor, then  $R_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{H} \cong R_{\mathbb{R}}$  (since this is a real form of  $R \otimes M_2(\mathbb{C}) \cong R$ ). In fact we also have

**THEOREM 1.8** ([20]). *The injective  $II_{\infty}$  factor has a unique real form.*

In the commutative case, there is no difference between antiautomorphisms and automorphisms. Thus we get the following classification theorem.

**THEOREM 1.9** ([4, Theorem 9.1]). *Commutative real  $C^*$ -algebras are classified by pairs consisting of a locally compact Hausdorff space  $X$  and a self-homeomorphism  $\tau$  of  $X$  satisfying  $\tau^2 = 1$ . The algebra associated to  $(X, \tau)$ , denoted  $C_0(X, \tau)$ , is  $\{f \in C_0(X) \mid f(\tau(x)) = \overline{f(x)} \forall x \in X\}$ .*

PROOF. If  $A$  is a commutative real  $C^*$ -algebra, then  $A_{\mathbb{C}} \cong C_0(X)$  for some locally compact Hausdorff space. The  $*$ -antiautomorphism  $\theta$  discussed above becomes a  $*$ -automorphism of  $A_{\mathbb{C}}$  (since the order of multiplication is immaterial) and thus comes from a self-homeomorphism  $\tau$  of  $X$  satisfying  $\tau^2 = 1$ . We recover  $A$  as

$$\begin{aligned} \{f \in A_{\mathbb{C}} \mid \sigma(f) = f\} &= \{f \in A_{\mathbb{C}} \mid \theta(f) = f^*\} \\ &= \{f \in C_0(X) \mid f(\tau(x)) = \overline{f(x)} \ \forall x \in X\}. \end{aligned}$$

In the other direction, given  $X$  and  $\tau$ , the indicated formula certainly gives a commutative real  $C^*$ -algebra.  $\square$

One could also ask about the classification of real AF algebras. This amounts to answering Problems 1.5 and 1.6 for complex AF algebras  $A$  (inductive limits of finite dimensional  $C^*$ -algebras). Since complex AF algebras are completely classified by  $K$ -theory ( $K_0(A)$  as an ordered group, plus the order unit if  $A$  is unital) [17], one would expect a purely  $K$ -theoretic solution. This was provided by Giordano [21], but the answer is considerably more complicated than in the complex case. Of course this is hardly surprising, since we already know that even the simplest noncommutative finite dimensional complex  $C^*$ -algebra,  $M_2(\mathbb{C})$ , has two distinct real structures. Giordano also showed that his invariant is equivalent to one introduced by Goodearl and Handelman [26]. We will not attempt to give the precise statement except to say that it involves all three of  $KO_0$ ,  $KO_2$ , and  $KO_4$ . (For example, one can distinguish the algebras  $M_2(\mathbb{R})$  and  $\mathbb{H}$ , both real forms of  $M_2(\mathbb{C})$ , by looking at  $KO_2$ .) Also, unlike the complex case, one usually has to deal with torsion in the  $K$ -groups.

For the rest of this paper, we will focus on the case of separable type I  $C^*$ -algebras, especially those that arise in representation theory. Recall that if  $A$  is a separable type I (complex)  $C^*$ -algebra, with primitive ideal space  $\text{Prim } A$  (equipped with the Jacobson topology), then the natural map  $\widehat{A} \rightarrow \text{Prim } A$ , sending the equivalence class of an irreducible representation  $\pi$  to its kernel  $\ker \pi$ , is a bijection, and enables us to put a  $T_0$  locally quasi-compact topology on  $\widehat{A}$ .

Suppose  $A$  is a complex  $C^*$ -algebra with a real form. As we have seen, that means  $A$  is equipped with a conjugate-linear  $*$ -automorphism  $\sigma$  with  $\sigma^2 = 1$ , or alternatively, with a linear  $*$ -antiautomorphism  $\theta$  with  $\theta^2 = 1$ . We can think of  $\theta$  as an isomorphism  $\theta: A \rightarrow A^{\text{op}}$  (where  $A^{\text{op}}$  is the opposite algebra, the same complex vector space with the same involution  $*$ , but with multiplication reversed) such that the composite  $\theta^{\text{op}} \circ \theta$  is the identity (from  $A$  to  $(A^{\text{op}})^{\text{op}} = A$ ).

Now let  $\pi$  be an irreducible  $*$ -representation of  $A$ . We can think of  $\pi$  as a map  $A \rightarrow \mathcal{B}(\mathcal{H})$ , and obviously,  $\pi$  induces a related map (which as a map of sets is exactly the same as  $\pi$ )  $\pi^{\text{op}}: A^{\text{op}} \rightarrow \mathcal{B}(\mathcal{H})^{\text{op}}$ . Composing with  $\theta$  and with the standard  $*$ -antiautomorphism  $\tau: \mathcal{B}(\mathcal{H})^{\text{op}} \rightarrow \mathcal{B}(\mathcal{H})$  (the “transpose map”) coming from the identification of  $\mathcal{B}(\mathcal{H})$  as the complexification of  $\mathcal{B}(\mathcal{H}_{\mathbb{R}})$ , we get a composite map

$$A \xrightarrow{\theta} A^{\text{op}} \xrightarrow{\pi^{\text{op}}} \mathcal{B}(\mathcal{H})^{\text{op}} \xrightarrow{\tau} \mathcal{B}(\mathcal{H}).$$

$\theta_*(\pi)$

One can also see that doing this twice brings us back where we started, so we have seen:

**PROPOSITION 1.10.** *If  $A$  is a complex  $C^*$ -algebra (for our purposes, separable and type I, though this is irrelevant here) with a real structure (given by a  $*$ -antiautomorphism  $\theta$  of period 2), then  $\theta$  induces an involution on  $\hat{A}$ .*

**PROOF.** The involution sends  $[\pi] \mapsto [\theta_*(\pi)]$ . To show that this is an involution, let’s compute  $\theta_*(\theta_*(\pi))$ . By definition, this is the composite

$$A \xrightarrow{\theta} A^{\text{op}} \xrightarrow{\theta_*(\pi)^{\text{op}}} \mathcal{B}(\mathcal{H})^{\text{op}} \xrightarrow{\tau} \mathcal{B}(\mathcal{H})$$

or

$$A \xrightarrow{\theta} A^{\text{op}} \xrightarrow{\theta} A \xrightarrow{\pi} \mathcal{B}(\mathcal{H}) \xrightarrow{\tau} \mathcal{B}(\mathcal{H})^{\text{op}} \xrightarrow{\tau} \mathcal{B}(\mathcal{H}),$$

$\theta_*(\pi)^{\text{op}}$

but  $\theta \circ \theta$  and  $\tau \circ \tau$  are each the identity, so this is just  $\pi$  again.  $\square$

Note that in the commutative case  $A = C_0(X)$ , the involution  $\theta_*$  on  $\hat{A}$  is just the original involution on  $X$ .

With these preliminaries out of the way, we can now begin to analyze the structure of (separable) real type I  $C^*$ -algebras. Some of this information is undoubtedly known to experts, but it is surprisingly hard to dig it out of the literature, so we will try give a complete treatment, without making any claims of great originality.

The one case which *is* easy to find in the literature concerns finite-dimensional real  $C^*$ -algebras, which are just semisimple Artinian algebras over  $\mathbb{R}$ . The interest in this case comes from the real group rings  $\mathbb{R}G$  of finite groups  $G$ , which are precisely of this type. A convenient reference for the real representation theory of finite groups is [64, §13.2]. A case which is not much harder is that of real representation theory of compact groups. In this case, the associated real  $C^*$ -algebra is infinite-dimensional in general, but splits as a  $(C^*$ -)direct

sum of (finite-dimensional) simple Artinian algebras over  $\mathbb{R}$ . This case is discussed in great detail in [1, Ch. 3], and is applied to connected compact Lie groups in [1, Ch. 6 and 7].

Recall from Corollary 1.4 that the irreducible representations of a real  $C^*$ -algebra  $A$  are of three types. How does this type classification relate to the involution of Proposition 1.10 on  $\widehat{A_{\mathbb{C}}}$ ? The answer (which for the finite group case appears in [64, §13.2]) is given as follows:

**THEOREM 1.11.** *Let  $A$  be a real  $C^*$ -algebra and let  $A_{\mathbb{C}}$  be its complexification. Let  $\pi$  be an irreducible representation of  $A$  (on a real Hilbert space  $\mathcal{H}$ ). If  $\pi$  is of real type, then we get an irreducible representation  $\pi_{\mathbb{C}}$  of  $A_{\mathbb{C}}$  on  $\mathcal{H}_{\mathbb{C}}$  by complexifying, and the class of this irreducible representation  $\pi_{\mathbb{C}}$  is fixed by the involution of Proposition 1.10. If  $\pi$  is of complex type, then  $\mathcal{H}$  can be made into a complex Hilbert space  $\mathcal{H}^c$  (whose complex dimension is half the real dimension of  $\mathcal{H}$ ) either via the action of  $\pi(A)'$  or via the conjugate of this action, and we get two distinct irreducible representations of  $A_{\mathbb{C}}$  on  $\mathcal{H}^c$  which are interchanged under the involution of Proposition 1.10 on  $\widehat{A_{\mathbb{C}}}$ . Finally, if  $\pi$  is of quaternionic type, then  $\mathcal{H}$  can be made into a quaternionic Hilbert space via the action of  $\pi(A)'$ . After tensoring with  $\mathbb{C}$ , we get a complex Hilbert space  $\mathcal{H}_{\mathbb{C}}$  whose complex dimension is twice the quaternionic dimension of  $\mathcal{H}$ , and we get an irreducible representation  $\pi_{\mathbb{C}}$  of  $A_{\mathbb{C}}$  on  $\mathcal{H}_{\mathbb{C}}$  whose class is fixed by the involution of Proposition 1.10.*

Now suppose further that  $A$  is separable and type I, so that  $\pi(A)$  contains the compact operators on  $\mathcal{H}$ , and in particular, there is an ideal  $\mathfrak{m}$  in  $A$  which maps onto the trace-class operators. Thus  $\pi$  has a well-defined “character”  $\chi$  on  $\mathfrak{m}$  in the sense of [29], and the representations  $\pi_{\mathbb{C}}$  of  $A_{\mathbb{C}}$  discussed above have characters  $\chi_{\mathbb{C}}$  on  $\mathfrak{m}_{\mathbb{C}}$ . When  $\pi$  is of real type,  $\chi_{\mathbb{C}}$  restricted to  $\mathfrak{m}$  is just  $\chi$  (and is real-valued). When  $\pi$  is of quaternionic type,  $\chi_{\mathbb{C}}$  restricted to  $\mathfrak{m}$  is  $\frac{\chi}{2}$ . When  $\pi$  is of complex type, the two complex irreducible extensions of  $\pi$  have characters on  $\mathfrak{m}$  which are non-real and which are complex conjugates of each other, and which add up to  $\chi$ .

**EXAMPLE 1.12.** Before giving the proof, it might be instructive to give some examples. First let  $A = C_{\mathbb{R}}^*(\mathbb{Z})$ , the free real  $C^*$ -algebra on one unitary  $u$ . The trivial representation  $u \mapsto 1$  is of real type and complexifies to the trivial representation of  $A_{\mathbb{C}} = C^*(\mathbb{Z})$ . Similarly the *sign* representation  $u \mapsto -1$  is of real type. The representation  $u \mapsto \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}$  on  $\mathbb{R}^2$  ( $\phi$  not a multiple of  $\pi$ ) is of complex type. Note that this representation is equivalent to the one given by

$u \mapsto \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}$  since these are conjugate under  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . This representation class corresponds to a pair of inequivalent irreducible representations of  $A_{\mathbb{C}} = C^*(\mathbb{Z})$  on  $\mathbb{C}$ , one given by  $u \mapsto e^{i\phi}$  and one given by  $u \mapsto e^{-i\phi}$ . The involution on  $\widehat{A_{\mathbb{C}}}$  sends one of these to the other.

Next let  $A = \mathbb{R}Q_8$ , the group ring of the quaternion group of order 8. This has a standard representation on  $\mathbb{H} \cong \mathbb{R}^4$  (sending the generators  $i, j, k \in Q_8$  to the quaternions with the same name) which is of quaternionic type. Complexifying gives two copies of the unique 2-dimensional irreducible complex representation of  $A_{\mathbb{C}}$ .

Note incidentally that  $D_8$ , the dihedral group of order 8, and  $Q_8$  have the same *complex* representation theory. Keeping track of the types of representations enables us to distinguish the two groups.

PROOF OF THEOREM 1.11. A lot of this is obvious, so we will just concentrate on the parts that are not. If  $\pi$  is of real type, its commutant is  $\mathbb{R}$  and its complexification  $\pi_{\mathbb{C}}$  has commutant  $\mathbb{C}$  and is thus irreducible. The class of  $\pi_{\mathbb{C}}$  is fixed by the involution, since for  $a \in A$ ,

$$\theta_*(\pi_{\mathbb{C}})(a) = \tau \circ \pi_{\mathbb{C}}^{\text{op}} \circ \theta(a) = \tau \circ \pi_{\mathbb{C}}^{\text{op}}(a^*) = \tau(\pi^{\text{op}}(a^*)) = (\pi(a)^t)^t = \pi(a).$$

If  $\pi$  is of complex type, we need to show that we get two distinct irreducible representations of  $A_{\mathbb{C}}$  which are interchanged under the involution on  $\widehat{A_{\mathbb{C}}}$ . If this were not the case, then  $\pi$  viewed as an irreducible representation on a complex Hilbert space  $\mathcal{H}^c$  (via the identification of  $\pi(A)'$  with  $\mathbb{C}$ ) would extend to an irreducible complex representation (let's call it  $\pi^c$ ) of  $A_{\mathbb{C}}$  which is isomorphic to  $\theta_*\pi^c$ . Now if  $a + ib \in A_{\mathbb{C}}$ , where  $a, b \in A$ , then  $\sigma(a + ib) = a - ib$  and  $\theta(a + ib) = a^* + ib^*$ . So  $\theta_*(\pi^c)(a + ib) = \tau \circ \pi_{\mathbb{C}}^{\text{op}}(a^* + ib^*) = \overline{\pi(a)} + i\overline{\pi(b)}$ , since for operators on  $\mathcal{H}^c$ ,  $\tau(T^*) = \overline{T}$ , the conjugate operator. The complexification of  $\mathcal{H}$  is canonically identified with  $\mathcal{H}^c \oplus \overline{\mathcal{H}^c}$ , so the complexification  $\pi_{\mathbb{C}}$  of  $\pi$  is thus identified with  $\pi^c \oplus \theta_*(\pi^c)$ . If this were isomorphic to  $\pi^c \oplus \pi^c$ , then its commutant would be isomorphic to  $M_2(\mathbb{C})$ . But the commutant of  $\pi_{\mathbb{C}}$  must be the complexification of the commutant of  $\pi$ , so this is impossible. If  $\pi$  is of quaternionic type, its commutant is isomorphic to  $\mathbb{H}$ , which complexifies to  $M_2(\mathbb{C})$ . That means the complexification of  $\pi$  has commutant  $M_2(\mathbb{C})$ , and thus the complexification of  $\pi$  is unitarily equivalent to a direct sum of two copies of an irreducible representation  $\pi_{\mathbb{C}}$  of  $A_{\mathbb{C}}$ . That the class of  $\pi_{\mathbb{C}}$  is fixed by the involution follows as in the real case.

Now let's consider the part about characters. If  $\pi$  is of real type,  $\pi_{\mathbb{C}}$  is its complexification and so the characters of  $\pi_{\mathbb{C}}$  and of  $\pi$  coincide on  $\mathfrak{m}$ . (Complexification of operators preserves traces.) If  $\pi$  is of quaternionic type, its complexification is equivalent to two copies of  $\pi_{\mathbb{C}}$ , so on  $\mathfrak{m}$ , the character  $\chi$  of  $\pi$  is the character of the complexification of  $\pi$  and so coincides with twice the character  $\chi_{\mathbb{C}}$  of  $\pi_{\mathbb{C}}$ , which is thus necessarily real-valued. Finally, suppose  $\pi$  is of complex type. If  $a \in \mathfrak{m}$ , then  $\theta_*(\pi_{\mathbb{C}})(a) = \overline{\pi(a)}$ , so we see in particular that the characters of  $\pi^c$  and of  $\theta_*(\pi^c)$  (on  $\mathfrak{m}$ ) are complex conjugates of one another, and add up to the character of  $\pi_{\mathbb{C}} \cong \pi^c \oplus \theta_*(\pi^c)$ . But complexification of an operator doesn't change its trace, so  $\pi_{\mathbb{C}}$  and  $\pi$  have the same character on  $\mathfrak{m}$ , and the characters of  $\pi^c$  and of  $\theta_*(\pi^c)$  add up to  $\chi$  on  $\mathfrak{m}$ .  $\square$

REMARK 1.13. One can also phrase the results of Theorem 1.11 in a way more familiar from group representation theory. Let  $A$  be a real  $C^*$ -algebra and let  $\pi$  be an irreducible representation of  $A_{\mathbb{C}}$  on a complex Hilbert space  $\mathcal{H}_{\mathbb{C}}$  such that the class of  $\pi$  is fixed under the involution of Proposition 1.10. Then  $\pi$  is associated to an irreducible representation of  $A$  of either real or quaternionic type. To tell which, observe that one of two possibilities holds. The first possibility is there is an  $A$ -invariant real structure on  $\mathcal{H}_{\mathbb{C}}$ , i.e.,  $\mathcal{H}_{\mathbb{C}}$  is the complexification of a real Hilbert space  $\mathcal{H}$  which is invariant under  $A$ , in which case  $\pi$  is of real type. This condition is equivalent to saying that there is a conjugate-linear map  $\varepsilon: \mathcal{H}_{\mathbb{C}} \rightarrow \mathcal{H}_{\mathbb{C}}$  commuting with  $A$  and with  $\varepsilon^2 = 1$ . ( $\mathcal{H}$  is just the  $+1$ -eigenspace of  $\varepsilon$ .)

The second possibility is that there is a conjugate-linear map  $\varepsilon: \mathcal{H}_{\mathbb{C}} \rightarrow \mathcal{H}_{\mathbb{C}}$  commuting with  $A$  and with  $\varepsilon^2 = -1$ . In this case if we let  $i$  act on  $\mathcal{H}_{\mathbb{C}}$  via the complex structure and let  $j$  act by  $\varepsilon$ , then since  $\varepsilon$  is conjugate-linear,  $i$  and  $j$  anticommute, and so we get an  $A$ -invariant structure of a quaternionic vector space on  $\mathcal{H}_{\mathbb{C}}$ , whose dimension over  $\mathbb{H}$  is half the complex dimension of  $\mathcal{H}_{\mathbb{C}}$ . In this case,  $\pi$  clearly has quaternionic type. This point of view closely follows the presentation in [10, II, §6].

The books [1] and [10] discuss the question of how one can tell the type (real, complex, or quaternionic) of an irreducible representation of a compact Lie group. In this case, one also has a criterion based on the value of the Frobenius-Schur indicator  $\int \chi(g^2) dg$ , which is 1 for representations of real type, 0 for representations of complex type, and  $-1$  for representations of quaternionic type. But since this criterion is based on tensor products for representations for groups, it doesn't seem to generalize to real  $C^*$ -algebras in general.



## 2. Real $C^*$ -algebras of continuous trace

We return now to the structure theory of (separable, say) real  $C^*$ -algebras of type I.

**DEFINITION 2.1.** Let  $A$  be a real  $C^*$ -algebra with complexification  $A_{\mathbb{C}}$ . We say  $A$  has *continuous trace* if  $A_{\mathbb{C}}$  has continuous trace in the sense of [13, §4.5], that is, if elements  $a \in (A_{\mathbb{C}})_+$  for which  $\pi \mapsto \text{Tr } \pi(a)$  is finite and continuous on  $\widehat{A_{\mathbb{C}}}$  are dense in  $(A_{\mathbb{C}})_+$ .

**THEOREM 2.2.** *Any non-zero postliminal real  $C^*$ -algebra (this is equivalent to being type I, even in the non-separable case — see [49, Ch. 6] or [61, §4.6]) has a non-zero ideal of continuous trace.*

**PROOF.** That  $A_{\mathbb{C}}$  has a non-zero ideal  $I$  of continuous trace is [13, Lemma 4.4.4]. So we need to show that  $I$  can be chosen to be  $\sigma$ -invariant, or equivalently, to show that  $\widehat{I}$  can be chosen invariant under the involution of Proposition 1.10. Simply observe that  $I + \sigma(I)$  is still a closed two-sided ideal and is clearly  $\sigma$ -invariant. Furthermore, it still has continuous trace since if  $a \in I_+$  and  $t_a: \pi \mapsto \text{Tr } \pi(a)$  is finite and continuous, then  $\pi \mapsto \text{Tr } \pi(\sigma(a)) = \text{Tr } \pi(\theta(a)) = \text{Tr } \theta_*(\pi)(a) = t_a \circ \theta_*(\pi)$  is also finite and continuous, so that  $\sigma(a)$  is also a continuous-trace element.  $\square$

**COROLLARY 2.3.** *Any non-zero postliminal real  $C^*$ -algebra has a composition series (possibly transfinite) with subquotients of continuous trace.*

**PROOF.** This follows by transfinite induction just as in the complex case.  $\square$

Because of Theorem 2.2 and Corollary 2.3, it is reasonable to focus special attention on real  $C^*$ -algebras with continuous trace. To such an algebra  $A$  (which we will assume is separable to avoid certain pathologies, such as the possibility that the spectrum might not be paracompact) is associated a *Real space*  $(X, \iota)$  in the sense of Atiyah [5], that is, a locally compact Hausdorff space  $X = \widehat{A_{\mathbb{C}}}$  and an involution  $\iota$  on  $X$  defined by Proposition 1.10. The problem then arises of classifying all the real continuous-trace algebras associated to a fixed Real space  $(X, \iota)$ . There is always a unique such commutative real  $C^*$ -algebra, given by Theorem 1.9.

When one considers noncommutative algebras,  $*$ -isomorphism is too fine for most purposes, and the most natural equivalence relation turns out to be *Morita equivalence*, which works for real  $C^*$ -algebras just as it does for complex  $C^*$ -algebras. Convenient references for the

theory of Morita equivalence (in the complex case) are [54, 53]. A Morita equivalence between real  $C^*$ -algebras  $A$  and  $B$  is given by an  $A$ - $B$  bimodule  $X$  with  $A$ -valued and  $B$ -valued inner products, satisfying a few simple axioms:

- (1)  $\langle x, y \rangle_A z = x \langle y, z \rangle_B$  and  $\langle a \cdot x, y \rangle_B = \langle x, a^* \cdot y \rangle_B$ ,  $\langle x \cdot b, y \rangle_A = \langle x, y \cdot b^* \rangle_A$  for  $x, y, z \in X$  and  $a \in A$ ,  $b \in B$ .
- (2) The images of the inner products are dense in  $A$  and in  $B$ .
- (3)  $\|\langle x, x \rangle_A\|_A^{1/2} = \|\langle x, x \rangle_B\|_B^{1/2}$  is a norm on  $X$ ,  $X$  is complete for this norm, and  $A$  and  $B$  act continuously on  $X$  by bounded operators.

The real continuous-trace algebras with spectrum  $(X, \iota)$  have been completely classified by Moutou [46] up to spectrum-fixing Morita equivalence, at least in the separable case. (Actually Moutou worked with graded  $C^*$ -algebras. See also [15, §3.3] for a translation into the ungraded case and the language we use here.)

First let us define the fundamental invariants.

**DEFINITION 2.4.** Let  $A$  be a real continuous-trace algebra with spectrum  $(X, \iota)$ . In other words  $X = \widehat{A_{\mathbb{C}}}$ , which is Hausdorff since  $A$  has continuous trace, and let  $\iota$  be the involution on  $X$  defined by Proposition 1.10. The *sign choice* of  $A$  is the map  $\alpha: X^\iota \rightarrow \{+, -\}$  attaching a  $+$  sign to fixed points of real type and a  $-$  sign to fixed points of quaternionic type. (Of course,  $\iota$  acts freely on  $X \setminus X^\iota$ , and the orbits of this action correspond to the pairs of conjugate representations of complex type.)

Note that if we give  $\{+, -\}$  the discrete topology, then it is easy to see that  $\alpha$  is continuous<sup>2</sup>, so it is constant on each connected component of  $X^\iota$ .

Incidentally, the name *sign choice* for this invariant comes from a physical application we will see in Section 3, where it is related to the signs of O-planes in string theory.

The other invariant of a (separable) real continuous-trace algebra is the *Dixmier-Douady invariant*. For a *complex* continuous-trace algebra with spectrum  $X$ , this is a class in  $H^2(X, \mathcal{T})$  (sheaf cohomology), where  $\mathcal{T}$  is the sheaf of germs of continuous  $\mathbb{T}$ -valued functions on  $X$ . We have a short exact sequence of sheaves

$$(2.1) \quad 0 \rightarrow \mathbb{Z} \rightarrow \mathcal{R} \rightarrow \mathcal{T} \rightarrow 1,$$

---

<sup>2</sup>One way to see this is to apply the part of Theorem 1.11 about characters. If  $e \in A_+$  is a local minimal projection near  $x \in X$ , then  $\text{Tr } \pi(e) = 1$  if  $\pi$  is close to  $x$  and  $\alpha(x) = +$  and  $\text{Tr } \pi(e) = 2$  if  $\pi$  is close to  $x$  and  $\alpha(x) = -$ .

where  $\mathcal{R}$  is the sheaf of germs of continuous real-valued functions, coming from the short exact sequence of abelian groups

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathbb{T} \rightarrow 1.$$

Since  $\mathcal{R}$  is a fine sheaf and thus has no higher cohomology, the long exact sequence in sheaf cohomology coming from (2.1) gives  $H^2(X, \mathcal{T}) \cong H^3(X, \mathbb{Z})$ , and indeed, the Dixmier-Douady invariant is usually presented as a class in  $H^3$ .

However, for purposes of dealing with *real* continuous-trace algebras, we need to take the involution  $\iota$  on  $X$  (and on  $\mathcal{T}$ ) into account. This will have the effect of giving a Dixmier-Douady invariant in an *equivariant* cohomology group  $H_\iota^2(X, \mathcal{T})$  defined by Moutuou [45], who denotes it  $HR^2(X, \mathcal{T})$  (with the  $\iota$  understood). The  $HR^\bullet$  groups are similar to, but not identical with, the  $\mathbb{Z}/2$ -equivariant cohomology groups  $H^\bullet(X; \mathbb{Z}/2, \mathcal{F})$  as defined in Grothendieck's famous paper [28, Ch. V]. The precise relationship in our situation is as follows:

**THEOREM 2.5.** *Let  $(X, \iota)$  be a second-countable locally compact Real space, i.e., space with an involution, and let  $\pi: X \rightarrow Y$  be the quotient map to  $Y = X/\iota$ . Then if  $\mathcal{T}$  is the sheaf of germs of  $\mathbb{T}$ -valued continuous functions on  $X$ , equipped with the involution induced by the involution  $(x, z) \mapsto (\iota(x), \bar{z})$  on  $X \times \mathbb{T}$ , then Moutuou's  $HR^2(X, \mathcal{T})$  coincides with  $H^2(Y, \mathcal{T}^\iota)$ , where  $\mathcal{T}^\iota$  is the induced sheaf on  $Y$ , i.e., the sheafification of the presheaf  $U \mapsto C(\pi^{-1}(U), \mathbb{T})^\iota$ . By [28, (5.2.6)], there is an edge homomorphism  $H_\iota^2(X, \mathcal{T}) \rightarrow H^2(X; \mathbb{Z}/2, \mathcal{T})$  (which is not necessarily an isomorphism).*

**PROOF.** In order to deal with quite general topological groupoids, Moutuou's definition of  $H_\iota^\bullet(X, \mathcal{F})$  in [45] uses simplicial spaces and a Čech construction. But in our situation,  $X$  and  $Y$  are paracompact and the groupoid structure on  $X$  is trivial, so by the equivariant analogue of [68, Theorem 1.1] and the isomorphism between Čech cohomology and sheaf cohomology for paracompact spaces [24, Théorème II.5.10.1], it reduces here to ordinary sheaf cohomology.  $\square$

Grothendieck's equivariant cohomology groups  $H^\bullet(X; \mathbb{Z}/2, \mathcal{F})$  are the derived functors of the equivariant section functor  $X \mapsto \Gamma(X, \mathcal{F})^\iota$ . Moutuou's groups are generally smaller. A few examples will clarify the notion, and also explain the difference between Grothendieck's functor and Moutuou's.

- (1) If  $\iota$  is trivial on  $X$ , the involution on  $\mathcal{T}$  is just complex conjugation, and  $H_\iota^\bullet(X, \mathcal{T})$  can be identified with  $H^\bullet(X, \mathcal{T}^\iota) = H^\bullet(X, \mathbb{Z}/2)$ . Note, for example, that if  $X$  is a single point,

then Grothendieck's  $H^\bullet(X; \mathbb{Z}/2, \mathcal{T})$  would be the group cohomology  $H^\bullet(\mathbb{Z}/2, \mathbb{T})$ , which is  $\mathbb{Z}/2$  in every even degree, whereas Moutuou's  $H_\iota^\bullet(\text{pt}, \mathcal{T})$  is just  $H^\bullet(\text{pt}, \mathbb{T}^\iota) = \mathbb{Z}/2$ , concentrated in degree 0.

- (2) If  $\iota$  acts freely, so that  $\pi: X \rightarrow Y$  is a 2-to-1 covering map,  $H_\iota^\bullet(X, \mathcal{T})$  can be identified with Grothendieck's

$$H^{\bullet+1}(X; \mathbb{Z}/2, \mathbb{Z}) \cong H^{\bullet+1}(Y, \underline{\mathbb{Z}})$$

for  $\bullet > 0$ , via the equivariant version of the long exact sequence associated to (2.1) and [28, Corollaire 3, p. 205]. Here  $\underline{\mathbb{Z}}$  is a locally constant sheaf obtained by dividing  $X \times \mathbb{Z}$  by the involution sending  $(x, n)$  to  $(\iota x, -n)$ .

DEFINITION 2.6. Now we can explain the definition of the real Dixmier-Douady invariant of a separable real  $C^*$ -algebra  $A$ . Without loss of generality, we can tensor  $A$  with  $\mathcal{K}_{\mathbb{R}}$ , which doesn't change the algebra up to spectrum-fixing Morita equivalence. Then  $A_{\mathbb{C}}$  becomes stable, and is locally, but not necessarily globally, isomorphic to  $C_0(X, \mathcal{K})$ . By paracompactness (here we use separability of  $A$ , which implies  $X$  is second countable and thus paracompact), there is a locally finite covering  $\{U_j\}$  of  $X$  such that  $A_{\mathbb{C}}$  is trivial over each  $\{U_j\}$ . We can also assume each  $U_j$  is  $\iota$ -stable. The trivializations of  $A_{\mathbb{C}}$  over the  $U_j$  give a Čech cocycle in  $H^1(\{U_j\}, \mathcal{PU})$ , given by the “patching data” over overlaps  $U_j \cap U_k$ . Here  $\mathcal{PU}$  is the sheaf of germs of  $PU$ -valued continuous functions, since  $PU$  is the automorphism group of  $\mathcal{K}$ . The image of this class in  $H^1(X, \mathcal{PU}) \cong H^2(X, \mathcal{T})$  is the complex Dixmier-Douady invariant. Here we use the long exact cohomology sequence associated to the sequence of sheaves

$$(2.2) \quad 1 \rightarrow \mathcal{T} \rightarrow \mathcal{U} \rightarrow \mathcal{PU} \rightarrow 1,$$

where again the middle sheaf is fine since the infinite unitary group (with the strong or weak operator topology) is contractible.

In our situation, there is a little more structure because  $A_{\mathbb{C}}$  was obtained by complexifying  $A$ . So we have the conjugation  $\sigma$  on  $A_{\mathbb{C}}$ , which induces the involution  $\iota$  on  $X$  and on the sheaves  $\mathcal{U}$ ,  $\mathcal{PU}$ , and  $\mathcal{T}$  over  $X$ . Furthermore, the cocycle of the patching data must be  $\iota$ -equivariant, and so defines the real Dixmier-Douady invariant, which is its coboundary in  $H_\iota^2(X, \mathcal{T})$ .

THEOREM 2.7 (Moutuou [46]). *The spectrum-fixing Morita equivalence classes of real continuous-trace algebras over  $(X, \iota)$  form a group (where the group operation comes from tensor product over  $X$ ) which*

is isomorphic to  $H^0(X^\iota, \mathbb{Z}/2) \oplus H^2(X, \mathcal{T})$  via the map sending an algebra  $A$  to the pair consisting of its sign choice and real Dixmier-Douady invariant (in the sense of Definitions 2.4 and 2.6).

REMARK 2.8. The formulation of this theorem in [46] looks rather different, for a number of reasons, though it is actually more general. For an explanation of how to translate it into this form, see [15, §3.3].

EXAMPLE 2.9. Here are three examples, that might be relevant for physical applications, that show how one computes the Brauer group of Theorem 2.7 in practice. In all cases we will take  $X$  to be a K3-surface (a smooth simply connected complex projective algebraic surface with trivial canonical bundle) and the involution  $\iota$  to be holomorphic (algebraic).

- (1) Suppose the involution  $\iota$  is holomorphic and free. In this case the quotient  $Y = X/\iota$  is an Enriques surface (with fundamental group  $\mathbb{Z}/2$ ) and  $\iota$  reverses the sign of a global holomorphic volume form. (See for example [47, §1].) There is no sign choice invariant since the involution is free, and thus all representations must be of complex type. The Dixmier-Douady invariant lives in (twisted) 3-cohomology of the quotient space  $Y$ . By Poincaré duality,  $H^3(Y, \mathbb{Z}) \cong H_1(Y, \mathbb{Z})$ , but since  $X$  is 1-connected, the classifying map  $Y \rightarrow B\mathbb{Z}/2 = \mathbb{RP}^\infty$  is a 2-equivalence (an isomorphism on  $\pi_1$  and surjection on  $\pi_2$ ) and induces an isomorphism on twisted  $H_1$ . So  $H^3(Y, \mathbb{Z}) \cong H_1(Y, \mathbb{Z}) \cong H_1(B\mathbb{Z}/2, \mathbb{Z}) \cong H_1^{\text{group}}(\mathbb{Z}/2, \mathbb{Z}) = 0$ . So the Dixmier-Douady invariant is always trivial in this case.
- (2) If  $\iota$  is a so-called Nikulin involution (see [44, 69]), then  $X^\iota$  consists of 8 isolated fixed points. Let  $Z = (X \setminus X^\iota)/\iota$ . By transversality, the complement  $X \setminus X^\iota$  of the fixed-point set is still simply connected, so  $\pi_1(Z) \cong \mathbb{Z}/2$  and the map  $Z \rightarrow B\mathbb{Z}/2$  is a 2-equivalence. We have  $H_{\iota,c}^2(X \setminus X^\iota, \mathcal{T}) \cong H_c^3(Z, \mathbb{Z})$ , and by Poincaré duality,  $H_c^3(Z, \mathbb{Z}) \cong H_1(Z, \mathbb{Z}) \cong H_1(B\mathbb{Z}/2, \mathbb{Z}) = 0$ . From the long exact sequence

$$(2.3) \quad H^1(X^\iota, \mathbb{Z}/2) = 0 \rightarrow H_{\iota,c}^2(X \setminus X^\iota, \mathcal{T}) \rightarrow H_\iota^2(X, \mathcal{T}) \rightarrow H^2(X^\iota, \mathcal{T}) = 0,$$

(see [24, Théorème II.4.10.1]) we see that  $H_\iota^2(X, \mathcal{T}) = 0$  and the Dixmier-Douady invariant is always trivial in this case. However, there are many possibilities for the sign choice since  $H^0(X^\iota, \mathbb{Z}/2) \cong (\mathbb{Z}/2)^8$ .

- (3) It is well known that there are K3-surfaces  $X$  with a holomorphic map  $f: X \rightarrow \mathbb{CP}^2$  that is a two-to-one covering branched over a curve  $C \subset \mathbb{CP}^2$  of degree 6 and genus 10. Such a surface  $X$  admits a holomorphic involution  $\iota$  having  $C$  as fixed-point set. We want to compute the Brauer group of real continuous-trace algebras over  $(X, \iota)$ . Since  $X^\iota = C$  is connected, there are only two possible sign choices, and algebras with sign choice  $-$  are obtained from those with sign choice  $+$  simply by tensoring with  $\mathbb{H}$ . So we may assume the sign choice on the fixed set is a  $+$ . The calculation of the possible Dixmier-Douady invariants is complicated and uses Theorem 2.5.

**THEOREM 2.10.** *Let  $X$  be a K3-surface and  $\iota$  a holomorphic involution on  $X$  with fixed set  $X^\iota = C$  a smooth projective complex curve of genus 10 and with quotient space  $Y = X/\iota = \mathbb{CP}^2$ . Then  $H_\iota^2(X, \mathcal{T}) = 0$ .*

**PROOF.** By Theorem 2.5,  $H_\iota^2(X, \mathcal{T}) \cong H^2(\mathbb{CP}^2, \mathcal{F})$ , where the sheaf  $\mathcal{F}$  is  $\mathbb{Z}/2$  over  $C$  and is locally isomorphic to  $\mathcal{T}$  over the complement. By [24, Théorème II.4.10.1], we obtain an exact sequence

$$(2.4) \quad \begin{aligned} H^1(C, \mathbb{Z}/2) &\rightarrow H_{\iota,c}^2(X \setminus C, \mathcal{T}) \rightarrow H_\iota^2(X, \mathcal{T}) \rightarrow H^2(C, \mathbb{Z}/2) \\ &\rightarrow H_{\iota,c}^3(X \setminus C, \mathcal{T}) \rightarrow H_\iota^3(X, \mathcal{T}) \rightarrow H^3(C, \mathbb{Z}/2) = 0. \end{aligned}$$

Note that  $(X \setminus X^\iota)/\iota \cong \mathbb{CP}^2 \setminus C$ . Thus in (2.4),  $H_{\iota,c}^j(X \setminus C, \mathcal{T}) \cong H_c^{j+1}(\mathbb{CP}^2 \setminus C, \underline{\mathbb{Z}}) \cong H_{3-j}(\mathbb{CP}^2 \setminus C, \underline{\mathbb{Z}})$ . Since  $C \subset \mathbb{CP}^2$  is a curve of degree 6, the map  $H^2(\mathbb{CP}^2, \mathbb{Z}) \rightarrow H^2(C, \mathbb{Z})$  induced by the inclusion is multiplication by 6, and we find from the long exact sequence

$$H^2(\mathbb{CP}^2, \mathbb{Z}) \xrightarrow{6} H^2(C, \mathbb{Z}) \rightarrow H_c^3(\mathbb{CP}^2 \setminus C, \mathbb{Z}) \rightarrow 0$$

that  $H_c^3(\mathbb{CP}^2 \setminus C, \mathbb{Z}) \cong H_1(\mathbb{CP}^2 \setminus C, \mathbb{Z}) \cong \mathbb{Z}/6$ . This implies that for  $j \leq 1$ ,  $H_j(\mathbb{CP}^2 \setminus C, \underline{\mathbb{Z}})$  will coincide with the  $\mathbb{Z}$ -homology of a lens space with fundamental group  $\mathbb{Z}/6$ , or with  $H_j^{\text{group}}(\mathbb{Z}/6, \underline{\mathbb{Z}}) = \mathbb{Z}/2$ ,  $j$  even, and 0,  $j$  odd. Hence (2.4) reduces to

$$0 \rightarrow H_\iota^2(X, \mathcal{T}) \rightarrow \mathbb{Z}/2 \xrightarrow{\delta} \mathbb{Z}/2 \rightarrow H_\iota^3(X, \mathcal{T}) \rightarrow 0,$$

and  $H_\iota^2(X, \mathcal{T})$  is either 0 or  $\mathbb{Z}/2$ , depending on whether the connecting map  $\delta$  is nontrivial or not.

To complete the calculation, we use Theorem 2.5. This identifies  $H_\iota^2(X, \mathcal{T})$  with  $I_2^{2,0}$  in the spectral sequence

$$I_2^{p,q} = H^p(Y, H^q(\mathbb{Z}/2, \mathcal{T})) \Rightarrow H^{p+q}(X; \mathbb{Z}/2, \mathcal{T})$$

of [28, Théorème 5.2.1]. We will examine this spectral sequence as well as the other one in that theorem,

$$\Pi_2^{p,q} = H^p(\mathbb{Z}/2, H^q(X, \mathcal{T})) \Rightarrow H^{p+q}(X; \mathbb{Z}/2, \mathcal{T}).$$

First consider  $\Pi_2^{\bullet, \bullet}$ . We have a short exact sequence of sheaves

$$(2.5) \quad 1 \rightarrow (\mathcal{T})_{X \setminus C} \rightarrow \mathcal{T} \rightarrow (\mathcal{T})_C \rightarrow 1,$$

and  $\iota$  acts trivially on  $C$  and freely on  $X \setminus C$ . Thus  $H^q(\mathbb{Z}/2, (\mathcal{T})_{X \setminus C}) = 0$  for  $q > 0$  [28, Corollaire 3, p. 205]. So from the long exact cohomology sequence derived from (2.5),  $H^q(\mathbb{Z}/2, \mathcal{T}) = H^q(\mathbb{Z}/2, (\mathcal{T})_C)$  is supported on  $C$  for  $q > 0$ . On  $C$ , the action of  $\iota$  is by complex conjugation, and so one easily sees that  $H^q(\mathbb{Z}/2, \mathcal{T}) = H^q(\mathbb{Z}/2, (\mathcal{T})_C) = (\mathbb{Z}/2)_C$  for  $q > 0$  even, 0 for  $q$  odd. So for  $q > 0$ ,  $\Pi_2^{p,q}$  vanishes for  $q$  odd and is  $H^p(C, \mathbb{Z}/2)$  for  $q$  even, which is  $\mathbb{Z}/2$  for  $p = 0$  or 2,  $(\mathbb{Z}/2)^{20}$  for  $p = 1$ , and 0 for  $p > 2$ . In particular,  $\Pi_2^{0,1} = 0$ , so  $d_2: \Pi_2^{0,1} \rightarrow \Pi_2^{2,0}$  vanishes and so the edge homomorphism  $H_\iota^2(X, \mathcal{T}) \rightarrow H^2(X; \mathbb{Z}/2, \mathcal{T})$  is injective. Furthermore,  $\Pi_2^{1,1} = 0$  and  $\Pi_2^{0,2} = \mathbb{Z}/2$ , so  $H^2(X; \mathbb{Z}/2, \mathcal{T})$  is finite and

$$|H^2(X; \mathbb{Z}/2, \mathcal{T})| \leq 2 \cdot |H_\iota^2(X, \mathcal{T})|.$$

Equality will hold if and only if the map  $d_3: \Pi_3^{0,2} = \mathbb{Z}/2 \rightarrow \Pi_3^{3,0} = H_\iota^3(X, \mathcal{T})$  is trivial.

Now consider the other spectral sequence  $\Pi_2^{\bullet, \bullet}$ . We have  $H^0(X, \mathcal{T}) = C(X, \mathbb{T})$ , which since  $X$  is simply connected fits into an exact sequence

$$(2.6) \quad 0 \rightarrow \mathbb{Z} \rightarrow C(X, \mathbb{R}) \rightarrow C(X, \mathbb{T}) \rightarrow 1.$$

Now in the exact sequence (2.1), the action of  $\mathbb{Z}/2$  is via a combination of the involution  $\iota$  on  $X$  and complex conjugation, which corresponds to multiplication by  $-1$  on  $\mathcal{R}$  and  $\mathbb{Z}$ . Thus as a  $\mathbb{Z}/2$  module, the group on the left in (2.6) is really  $\underline{\mathbb{Z}}$ ,  $\mathbb{Z}$  with the non-trivial action. On the  $q = 0$  row, we use equation (2.6) and the fact that higher cohomology of a finite group with coefficients in a real vector space has to vanish to obtain that

$$\Pi_2^{p,0} = H^p(\mathbb{Z}/2, C(X, \mathbb{T})) \cong H^{p+1}(\mathbb{Z}/2, \underline{\mathbb{Z}}) \cong \begin{cases} 0, & p \text{ odd}, \\ \mathbb{Z}/2, & p \text{ even} > 0. \end{cases}$$

For  $q > 0$ , we know that  $H^q(X, \mathcal{T}) \cong H^{q+1}(X, \mathbb{Z})$ , which will be nonzero (and torsion-free) only for  $q = 1$  and  $q = 3$ . Again, since the action on  $\mathbb{Z}/2$  on the constant sheaf  $\mathbb{Z}$  in (2.1) is by multiplication by  $-1$ , the action of  $\mathbb{Z}/2$  on  $H^3(X, \mathcal{T}) \cong H^4(X, \mathbb{Z}) \cong \mathbb{Z}$  is by multiplication by  $-1$ . The case of  $H^1(X, \mathcal{T}) \cong H^2(X, \mathbb{Z}) \cong \mathbb{Z}^{22}$  is more complicated because we also have the action of  $\iota$  on  $H^2(X, \mathbb{Z})$ , which has fixed set of rank 1 [47, p. 595]. So we need to determine the

structure of  $H^2(X, \mathbb{Z})$  as a  $\mathbb{Z}/2$ -module. The action of  $\iota$  on  $H^2$  has to respect the intersection pairing, with respect to which  $H^2(X, \mathbb{Z})$  splits (non-canonically) as  $E_8 \oplus E_8 \oplus H \oplus H \oplus H$ , where  $E_8$  is the  $E_8$  lattice and  $H$  is a hyperbolic plane ( $\mathbb{Z}^2$  with form given by  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ). Since  $H^2(X, \mathbb{Z})^\iota \cong \mathbb{Z}$ , one can quickly see that the only possibility is that  $\iota$  acts by  $-1$  on both  $E_8$  summands and on two of the  $H$  summands, and interchanges the generators of the other  $H$  summand. Our action here of  $\mathbb{Z}/2$  is reversed from this, so as  $\mathbb{Z}/2$ -module,  $H^1(X, \mathcal{T}) \cong \mathbb{Z}^{20} \oplus H$ . Since  $H^p(\mathbb{Z}/2, \mathbb{Z})$  is non-zero only for  $p$  even and  $H^p(\mathbb{Z}/2, H) = 0$  for  $p > 0$  (by simple direct calculation, or else by Shapiro's Lemma, since  $H$  as a  $\mathbb{Z}/2$ -module is induced from  $\mathbb{Z}$  as a module for the trivial group), we find that  $\Pi_2^{1,1} = H^1(\mathbb{Z}/2, \mathbb{Z}^{20} \oplus H) = 0$ . Since we already computed that  $\Pi_2^{0,2} = 0$  and  $\Pi_2^{2,0} = \mathbb{Z}/2$ , we see that  $|H^2(X; \mathbb{Z}/2, \mathcal{T})| \leq 2$ . It will be 0 only if  $d_2: \Pi_2^{0,1} \rightarrow \Pi_2^{2,0}$  is non-zero. Putting everything together, we finally see that the only possibilities for the two spectral sequences are as in Figures 1 and 2. Comparing the (dotted) diagonal lines of total degrees 2 and 3 in the two figures, we conclude that  $H_\iota^2(X, \mathcal{T})$  and  $H_\iota^3(X, \mathcal{T})$  must both vanish.  $\square$

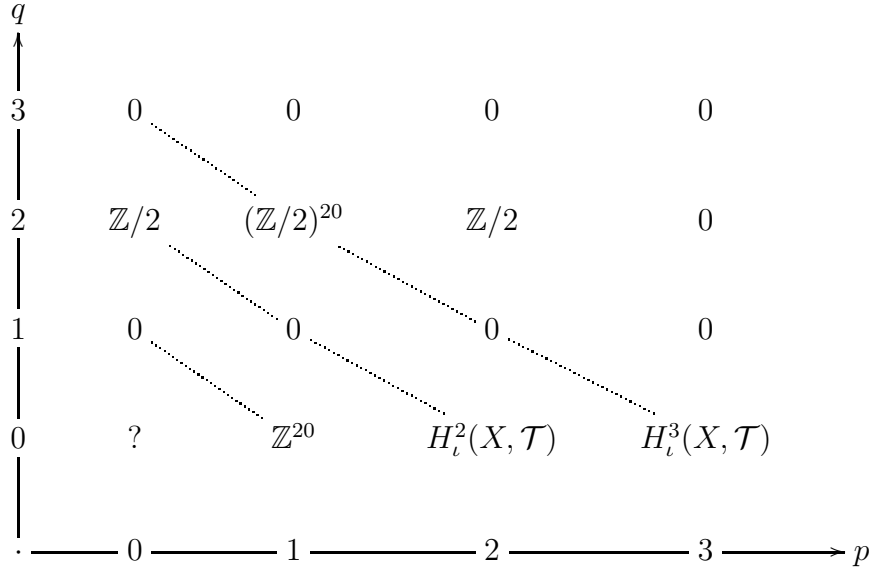
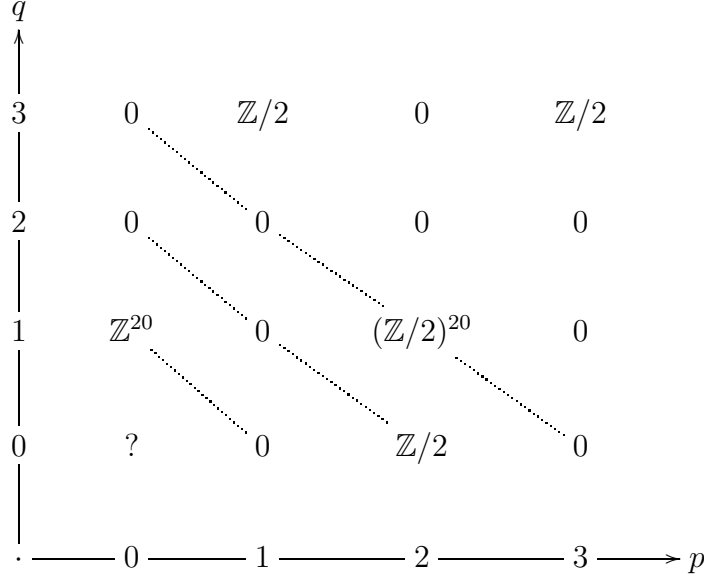


FIGURE 1. The first Grothendieck spectral sequence  $I_2^{\bullet, \bullet}$




 FIGURE 2. The second Grothendieck spectral sequence  $\mathrm{II}_2^{\bullet,\bullet}$ 

### 3. $K$ -Theory and Applications

In this last section, we will briefly discuss the (topological)  $K$ -theory of real  $C^*$ -algebras, and explain some key applications to manifolds of positive scalar curvature and to orientifold string theories in physics. We should mention that other physical applications have appeared in the theory of topological insulators in condensed matter theory [30, 36], though we will not go into this area here. Along the way, connections will show up with representation theory via the real Baum-Connes conjecture.

The (topological)  $K$ -theory of real  $C^*$ -algebras is of course a special case of topological  $K$ -theory of real Banach algebras. As such it has all the usual properties, such as homotopy invariance and Bott periodicity of period 8. A convenient reference is [63].

A nice feature of the  $K$ -theory of real continuous-trace  $C^*$ -algebras is that it unifies all the variants of topological  $K$ -theory (for spaces) that have appeared in the literature. This includes of course real  $K$ -theory  $KO$ , complex  $K$ -theory  $K$ , and symplectic  $K$ -theory  $KSp$ , but also Atiyah's "Real"  $K$ -theory  $KR$  [5], Dupont's symplectic analogue of  $KR$  [16], sometimes called  $KH$ , and the self-conjugate  $K$ -theory  $KSC$  of Anderson and Green [27, 3].  $KR^\bullet(X, \tau)$  is the topological  $K$ -theory of the commutative real  $C^*$ -algebra  $C_0(X, \tau)$  of Theorem 1.9.

$KSC^\bullet(X)$  is  $KR^\bullet(X \times S^1)$ , where  $S^1$  is given the (free) antipodal involution [5, Proposition 3.5]. In addition, the  $K$ -theory of a stable real continuous-trace with a sign choice (but vanishing Dixmier-Douady invariant) is “ $KR$ -theory with a sign choice” as defined in [14], and the  $K$ -theory of a stable real continuous-trace with no sign choice but a nontrivial Dixmier-Douady invariant is what has generally been called “twisted  $K$ -theory” (of either real or complex type, depending on the types of the irreducible representations of the algebra). See [55, 57] for some of the original treatments, as well as [6, 34] for more modern approaches.

**3.1. Positive scalar curvature.** A first area where real  $C^*$ -algebras and their  $K$ -theory plays a significant role is the classification of manifolds of positive scalar curvature. The first occurrence of real  $C^*$ -algebras in this area is implicit in an observation of Hitchin [31], that if  $M$  is a compact Riemannian spin manifold of dimension  $n$  with positive scalar curvature, then the  $KO_n$ -valued index of the Dirac operator on  $M$  has to vanish. For  $n$  divisible by 4, this observation was not new and goes back to Lichnerowicz [40], but for  $n \equiv 1, 2 \pmod{2}$ , a new torsion obstruction shows up that cannot be “seen” without real  $K$ -theory.

The present author observed that there is a much more extensive obstruction theory when  $M$  is not simply connected. Take the fundamental group  $\pi$  of  $M$ , a countable discrete group. Complete the real group ring  $\mathbb{R}\pi$  in its greatest  $C^*$ -norm to get the real group  $C^*$ -algebra  $A = C_{\mathbb{R}}^*(\pi)$ . (Alternatively, one could use the reduced real group  $C^*$ -algebra  $A_r = C_{\mathbb{R},r}^*(\pi)$ , the completion of the group ring for its left action on  $L^2(\pi)$ . For present purposes it doesn’t much matter.) Coupling the Dirac operator on  $M$  to the universal flat  $C_{\mathbb{R}}^*(\pi)$ -bundle  $\widetilde{M} \times_{\pi} A$  over  $M$ , one gets a Dirac index with values in  $KO_n(A)$ , which must vanish if  $M$  has positive scalar curvature. Thus we have a new source of obstructions to positive scalar curvature.

As shown in [56, 58], this  $KO_n(A)$ -valued index obstruction can be computed to be  $\mu \circ f_*(\alpha_M)$ , where  $\alpha_M \in KO_n(M)$  is the “Atiyah orientation” of  $M$ , i.e., the  $KO$ -fundamental class defined by the spin structure,  $f: M \rightarrow B\pi$  is a classifying map for the universal cover  $\widetilde{M} \rightarrow M$ , and  $\mu: KO_n(B\pi) \rightarrow KO_n(A)$  is the “real assembly map,” closely related to the Baum-Connes assembly map in [7].<sup>3</sup>

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<sup>3</sup>The relationship is this. Let  $\underline{E}\pi$  denote the universal proper  $\pi$ -space and let  $E\pi$  denote the universal free  $\pi$ -space. These coincide if and only if  $\pi$  is torsion-free. The Baum-Connes assembly map is defined on  $KO_{\bullet}^{\pi}(\underline{E}\pi)$  whereas our map is defined on  $KO_{\bullet}^{\pi}(E\pi) = KO_n(B\pi)$ . Since  $E\pi$  is a proper  $\pi$ -space, there is a canonical  $\pi$ -map  $E\pi \rightarrow \underline{E}\pi$  (unique up to equivariant homotopy) and so our  $\mu$

In [58, Theorem 2.5], I showed that for  $\pi$  finite, the image of the reduced assembly map  $\mu$  (what one gets after pulling out the contribution from the trivial group, i.e., the Lichnerowicz and Hitchin obstructions) is precisely the image in  $KO_\bullet(\mathbb{R}\pi)$  of the 2-torsion in  $KO_\bullet(\mathbb{R}\pi_2)$ ,  $\pi_2 \subseteq \pi$  a Sylow 2-subgroup. This lives in degrees 1 and 2 mod 4 and comes from the irreducible representations of  $\pi_2$  of real and quaternionic type, in the sense that we explained in Theorem 1.11. So far the obstructions detected by  $\mu$  are the only known obstructions to positive scalar curvature on closed spin manifolds of dimension  $> 4$  with finite fundamental group.

The problem of existence or non-existence of positive scalar curvature on a spin manifold can be split into two parts, one “stable” and one “nonstable.” Stability here refers to taking the product with enough copies of a “Bott manifold”  $Bt^8$ , a simply connected closed Ricci-flat 8-manifold representing the generator of Bott periodicity. Since index obstructions in  $K$ -theory of real  $C^*$ -algebras live in groups which are periodic mod 8, stabilizing the problem by crossing with copies of  $Bt^8$  compensates for this by introducing 8-periodicity on the geometric side. Indeed it was shown in [60] that the stable conjecture ( $M \times Bt^k$  admits a metric of positive scalar curvature for sufficiently large  $k$  if and only if  $\mu \circ f_*(\alpha_M)$  vanishes) holds when  $\pi$  is finite. Stolz has extended this theorem as follows: for a completely general closed spin manifold  $M^n$  with fundamental group  $\pi$ ,  $M \times Bt^k$  admits a metric of positive scalar curvature for sufficiently large  $k$  if and only if  $\mu \circ f_*(\alpha_M)$  vanishes, provided that the real Baum-Connes conjecture (bijectivity of the Baum-Connes assembly map  $\mu: KO_\bullet^\pi(E\pi) \rightarrow KO_\bullet(C_{\mathbb{R},r}^*)$ ) holds for  $\pi$ . In fact, Baum-Connes can be weakened here in two ways — one only needs injectivity of  $\mu$ , not surjectivity, and one can replace  $C_{\mathbb{R},r}^*$  by the full  $C^*$ -algebra  $C_{\mathbb{R}}^*$ . Since the full  $C^*$ -algebra surjects onto the reduced  $C^*$ -algebra, injectivity of the assembly map for the full  $C^*$ -algebra is a weaker condition. Sketches of Stolz’s theorem may be found in [65, 66], though unfortunately the full proof of this was never written up.

Since the real version of the Baum-Connes conjecture has just been seen to play a fundamental role here, it is worth remarking that the real and complex versions of the Baum-Connes conjecture are actually equivalent [8, 62]. Thus the real Baum-Connes conjecture holds in the huge number of cases where the complex Baum-Connes conjecture has been verified.

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factors through the Baum-Connes assembly map, and agrees with it if  $\pi$  has no torsion.

**3.2. Representation theory.** Since we have already mentioned the real Baum-Connes conjecture, it is worth mentioning that this, as well as the general theory of real  $C^*$ -algebras, has some relevance to representation theory. Suppose  $G$  is a locally compact group (separable, say, but not necessarily discrete). The real group  $C^*$ -algebra  $C_{\mathbb{R}}^*(G)$  is the completion of the real  $L^1$ -algebra (the convolution algebra of real-valued  $L^1$  functions on  $G$ ) for the maximal  $C^*$ -algebra norm. Obviously this defines a canonical real structure on the complex group  $C^*$ -algebra  $C^*(G)$ , and similarly we have  $C_{\mathbb{R},r}^*(G)$  inside the reduced  $C^*$ -algebra  $C_r^*(G)$ . Computing the structure of  $C_{\mathbb{R}}^*(G)$  or of  $C_{\mathbb{R},r}^*(G)$  gives us more information than just computing the structure of their complexifications. For instance, it gives us the type classification of the representations, as we saw in Theorem 1.11 and Example 1.12. The real Baum-Connes conjecture, when it's known to hold, gives us at least partial information on the structure of  $C_{\mathbb{R},r}^*(G)$  (its  $K$ -theory).

Here are some simple examples (where the real structure is not totally uninteresting) to illustrate these ideas.

EXAMPLE 3.1. Let  $G = SU(2)$ . As is well known, this has (up to equivalence) irreducible complex representation of each positive integer dimension. It is customary to parameterize the representations  $V_k$  by the value of the “spin”  $k = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$  (this is the highest weight divided by the unique positive root), so that  $\dim V_k = 2k + 1$ . The character  $\chi_k$  of  $V_k$  is given on a maximal torus by  $e^{i\theta} \mapsto \frac{\sin(2k+1)\theta}{\sin \theta}$ , which is real-valued, and thus all the representations must have real or quaternionic type. In fact,  $V_k$  is of real type if  $k$  is an integer and is of quaternionic type if  $k$  is a half-integer. (That's because  $V_1$  is the complexification of the covering map  $SU(2) \rightarrow SO(3)$ , while  $V_{1/2}$  acts on the unit quaternions, and all the other representations can be obtained from these by taking tensor products and decomposing. A tensor product of real representations is real, and a tensor product of a real representation with a quaternionic one is quaternionic.) Indeed if one computes the Frobenius-Schur indicator  $\int_G \chi_k(g^2) dg$  for  $V_k$  using the Weyl integration formula, one gets

$$\begin{aligned} \int_G \chi_k(g^2) dg &= \frac{1}{\pi} \int_0^{2\pi} \left( e^{4ki\theta} + e^{4(k-1)i\theta} + \dots + e^{-4ki\theta} \right) \sin^2 \theta d\theta \\ &= \frac{1}{4\pi} \int_0^{2\pi} \left( e^{4ki\theta} + e^{4(k-1)i\theta} + \dots + e^{-4ki\theta} \right) \left( 2 - e^{2i\theta} - e^{-2i\theta} \right) d\theta. \end{aligned}$$

If  $k$  is an integer, we get

$$\frac{1}{4\pi} \int_0^{2\pi} \left( (\dots + e^{4i\theta}) + 1 + (e^{-4i\theta} + \dots) \right) \left( 2 - e^{2i\theta} - e^{-2i\theta} \right) d\theta = 1,$$

where the terms in small parentheses are missing if  $k = 0$ , while if  $k$  is a half-integer, we get

$$\frac{1}{4\pi} \int_0^{2\pi} \left( \cdots + e^{2i\theta} + e^{-2i\theta} + \cdots \right) \left( 2 - e^{2i\theta} - e^{-2i\theta} \right) d\theta = -1.$$

This confirms the type classification we gave earlier.

Thus  $C_{\mathbb{R}}^*(G) \cong \bigoplus_{k \in \mathbb{N}} M_{2k+1}(\mathbb{R}) \oplus \bigoplus_{k=\frac{1}{2}, \frac{3}{2}, \dots} M_{k+\frac{1}{2}}(\mathbb{H})$ . In particular, we see that  $KO_{\bullet}(C_{\mathbb{R}}^*(G)) \cong (KO_{\bullet})^{\infty} \oplus (KSp_{\bullet})^{\infty}$ , and in degrees 1 and 2 mod 4, this is an infinite direct sum of copies of  $\mathbb{Z}/2$ , whereas the torsion-free contributions appear only in degrees divisible by 4. Conversely, if one had some independent method of computing  $KO_{\bullet}(C_{\mathbb{R}}^*(G))$ , it would immediately tell us that  $G$  has no irreducible representations of complex type, and infinitely many representations of both real and of quaternionic type.

EXAMPLE 3.2. Let  $H$  be the compact group  $\mathbb{T} \cup j\mathbb{T}$ , where  $j$  is an element with  $j^2 = -1$ ,  $jzj^{-1} = \bar{z}$  for  $z \in \mathbb{T}$ . This is a nonsplit extension of  $\text{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2$  by  $\mathbb{T}$  and is secretly the maximal compact subgroup of  $W_{\mathbb{R}}$ , the Weil group of the reals (which splits as  $\mathbb{R}_+^{\times} \times H$ ). The induced action of  $j$  on  $\widehat{\mathbb{T}} = \mathbb{Z}$  sends  $n \mapsto -n$ . So the Mackey machine tells us that the irreducible complex representations of  $H$  are the following:

- (1) two one-dimensional representations  $\chi_0^{\pm}$  which are trivial on  $\mathbb{T}$  and send  $j \mapsto \pm 1$ . These representations are obviously of real type.
- (2) a family  $\pi_n = \text{Ind}_{\mathbb{T}}^H \sigma_n$ ,  $n \in \mathbb{Z} \setminus \{0\}$  of two-dimensional representations, where  $\sigma_n(z) = z^n$ ,  $z \in \mathbb{T}$ . These representations are all of quaternionic type since they come from complexifying the representation  $H \rightarrow \mathbb{H}^{\times}$  given by  $z \mapsto z^n$ ,  $j \mapsto j$ .

We immediately conclude that  $C_{\mathbb{R}}^*(H) \cong \mathbb{R} \oplus \mathbb{R} \oplus (\mathbb{H})^{\infty}$ . Thus  $KO_{\bullet}(C_{\mathbb{R}}^*(H))$  is elementary abelian of rank 2 in degrees 1 and 2 mod 8, and is  $(\mathbb{Z}/2)^{\infty}$  in degrees 5 and 6 mod 8. Again, if we had an independent way to compute  $KO_{\bullet}(C_{\mathbb{R}}^*(H))$ , it would tell us about the types of the representations.

EXAMPLE 3.3. A slightly more interesting example is  $G = SL(2, \mathbb{C})$ , a simple complex Lie group with  $K = SU(2)$  as maximal compact subgroup. The reduced dual of  $G$  is Hausdorff, and the complex reduced  $C^*$ -algebra  $C_r^*(G)$  is a stable continuous-trace algebra with trivial Dixmier-Douady invariant, i.e., it is Morita equivalent to  $C_0(\widehat{G}_r)$ . All the irreducible complex representations of  $C_r^*(G)$  are principal series representations, and all unitary principal series are irreducible. Thus we see that  $C_r^*(G)$  is Morita equivalent to  $C_0(\widehat{M}/W)$ , where  $M$

is a Cartan subgroup, which we can take to be  $\mathbb{C}^\times$ , and  $W = \{\pm 1\}$  is the Weyl group, which acts on  $\widehat{\mathbb{C}^\times} \cong \mathbb{Z} \times \mathbb{R}$  by  $-1$  (on both factors). So  $C_r^*(G)$  is Morita equivalent to  $C_0([0, \infty)) \oplus \bigoplus_{n \geq 1} C_0(\mathbb{R})$ , with  $[0, \infty) = (\{0\} \times \mathbb{R})/W$ . (For all of this one can see [18, V] or [50], for example.) The complex Baum-Connes map gives an isomorphism  $K_\bullet^G(G/K) \cong R(K) \otimes K_{\bullet-3} \xrightarrow{\mu} K_\bullet(C_r^*(G))$ , sending the generator of the representation ring  $R(K)$  associated to  $V_k$  (in the notation of Example 3.1) to the generator of the  $\mathbb{Z}$  summand in  $K_0(C_r^*(G))$  associated to the principal series with discrete parameter  $\pm(2k+1)$ . There is no contribution to  $K_0(C_r^*(G))$  from the spherical principal series (corresponding to the fixed point  $n=0$  of  $W$  on  $\mathbb{Z}$ ) since  $\mathbb{R}/\{\pm 1\} \cong [0, \infty)$  is properly contractible.

Now let's analyze the real structure. We can start by looking at  $K$ -types. The group  $SL(2, \mathbb{C})$  is the double cover of the Lorentz group  $SO(3, 1)_0$ . Representations that descend to  $SO(3, 1)_0$  must have  $K$  types that factor through  $SU(2) \rightarrow SO(3)$ , and so have integral spin. All integral spin representations have real type, so these representations are also of real type, at least when restricted to  $K$ . The genuine representations of  $SL(2, \mathbb{C})$  that do not descend to  $SO(3, 1)_0$  must have  $K$ -types with half-integral spin, and these representations are of quaternionic type, at least when restricted to  $K$ . There is one principal series which is obviously of real type, namely the "0-point" of the spherical principal series, since this representation is simply  $\text{Ind}_B^G 1$ , where  $B$  is a Borel subgroup. Since the trivial representation of  $B$  is of real type, we get a real form for the complex induced representation by using induction with real Hilbert spaces instead. And thus we get an irreducible real representation on  $L_{\mathbb{R}}^2(G/B) \cong L_{\mathbb{R}}^2(K/\mathbb{T})$ . In fact the other spherical principal series can be realized on this same Hilbert space (see [18, p. 261]) so they, too, are of real type. But this method won't work for other characters of  $B$  since none of the other one-dimensional unitary characters of  $\mathbb{C}^\times$  are of real type. If we look at a principal series representation of  $G$  with discrete parameter  $\pm n \in \mathbb{Z}$ , its restriction to  $K$  can be identified with  $\text{Ind}_{\mathbb{T}}^K \chi_n$ , which by Frobenius reciprocity contains  $V_k$  with multiplicity equal to the multiplicity of  $\chi_n$  in  $V_k$ . This is 0 if  $2k$  and  $n$  have opposite parity or if  $|n| > 2k$ , and is 1 otherwise. So this principal series representation has all its  $K$ -types of multiplicity 1 and has real (resp., quaternionic) type when restricted to  $K$  provided  $n$  is even (odd).

We can analyze things in more detail by seeing what the involution (of Proposition 1.10) on  $\widehat{G}$  does to the principal series. Clearly it sends  $\text{Ind}_B^G \chi$  to  $\text{Ind}_B^G \overline{\chi}$ , if  $\chi$  is a one-dimensional representation of  $M$

viewed as a representation of  $B$ . But since  $W = \{\pm 1\}$ ,  $\bar{\chi} = w \cdot \chi$ , for  $w$  the generator of  $W$ , and we get an equivalent representation. Thus *the involution on  $\widehat{G}_r$  is trivial*. One can also check this very easily by observing that all the characters of  $G$  are real-valued. (See for example [70, Theorem 5.5.3.1], where again the key fact for us is that  $w \cdot \chi = \bar{\chi}$ .) J. Adams has studied this property in much greater generality and proved:

**THEOREM 3.4** (Adams [2, Theorem 1.8]). *If  $G$  is a connected reductive algebraic group over  $\mathbb{R}$  with maximal compact subgroup  $K$ , if  $-1$  lies in the Weyl group of the complexification of  $G$ , and if every irreducible representation of  $K$  is of real or quaternionic type, then every unitary representation of  $G$  is also of real or quaternionic type.*

Thus we know that the involution on  $\widehat{G}_r$  is trivial and that  $C_r^*(G)$  is a real  $C^*$ -algebra of continuous trace, with spectrum a countable union of contractible components, all but one of which are homeomorphic to  $\mathbb{R}$ , with the exceptional component homeomorphic to  $[0, \infty)$ . The real Dixmier-Douady invariants must all vanish since  $H^2(\widehat{G}_r, \mathbb{Z}/2) = 0$ . The sign choice invariants are now determined by the  $K$ -types, since all the  $K$ -types have multiplicity one and thus a representation of  $G$  of real (resp., quaternionic) type must have all its  $K$ -types of the same type. Putting everything together, we see that we have proved the following theorem:

**THEOREM 3.5.** *The reduced real  $C^*$ -algebra of  $SL(2, \mathbb{C})$  is a stable real continuous-trace algebra, Morita equivalent to*

$$C_0^{\mathbb{R}}([0, \infty)) \oplus \bigoplus_{n>0 \text{ even}} C_0^{\mathbb{R}}(\mathbb{R}) \oplus \bigoplus_{n>0 \text{ odd}} C_0^{\mathbb{H}}(\mathbb{R}).$$

Schick's proof in [62] that the complex Baum-Connes conjecture implies the real Baum-Connes conjecture is stated only for discrete groups, but it goes over without difficulty to general locally compact groups, at least in the case of trivial coefficients. Since the complex Baum-Connes conjecture (without coefficients) is known for all connected reductive Lie groups [71, 38], the real Baum-Connes map is an isomorphism  $KO_{\bullet}^G(G/K) \xrightarrow{\mu} KO_{\bullet}(C_{\mathbb{R},r}^*(G))$ . This by itself gives some information on the real structure of the various summands in  $C_{\mathbb{R},r}^*(G)$ . Since  $G/K$  has a  $G$ -invariant spin structure in this case, by the results of [35, §5],  $KO_{\bullet}^G(G/K) \cong KO_{\bullet}^K(G/K) \cong KO_{\bullet}^K(\mathfrak{p})$ , which by equivariant Bott periodicity is  $KO_{\bullet-3}(C_{\mathbb{R}}^*(K))$ , which we computed in Example 3.1. (Here  $\mathfrak{p}$  is the orthogonal complement to the Lie algebra of  $K$  inside the Lie algebra of  $G$ . In this case,  $\mathfrak{p}$  is isomorphic as a  $K$ -module

to the adjoint representation of  $K$ .) On the other side of the isomorphism, we have  $KO_{\bullet}(C_0^{\mathbb{R}}(\mathbb{R})) \cong KO^{-\bullet}(\mathbb{R}) \cong KO^{-\bullet-1}(\text{pt}) \cong KO_{\bullet+1}$ , and similarly  $KO_{\bullet}(C_0^{\mathbb{H}}(\mathbb{R})) \cong KSp^{-\bullet}(\mathbb{R}) \cong KO^{-\bullet-5}(\text{pt}) \cong KO_{\bullet+5}$ . So the torsion-free summands in  $KO_{\bullet}(C_{\mathbb{R},r}^*(G))$  are all in degrees 3 mod 4. Since there are no torsion-free summands in  $KO_{\bullet}(C_{\mathbb{R},r}^*(G))$  in degrees 1 mod 4, we immediately conclude that no unitary principal series (except perhaps for the spherical principal series, which don't contribute to the  $K$ -theory) are of complex type, and that there are infinitely many lines of principal series of both real and quaternionic type. This is a large part of Theorem 3.5, and is not totally trivial to check directly.

One interesting feature of the Baum-Connes isomorphism is the degree shift. Since  $KO_{\bullet}^G(G/K) \cong KO_{\bullet-3}(C_{\mathbb{R}}^*(K))$ , while  $KO_{\bullet}(C_{\mathbb{R},r}^*(G))$  is a sum of copies of  $KO_{\bullet+1}$ , associated to principal series with  $K$ -types of integral spin and  $KO_{\bullet+5}$ , associated to principal series with  $K$ -types of half-integral spin, we see that representations of  $K$  of integral spin on the left match with those of half-integral spin on the right, and those of half-integral spin on the left match with those of integral spin on the right. This is due to the “ $\rho$ -shift” in Dirac induction. If we were to replace  $SL(2, \mathbb{C})$  by the adjoint group  $G = PSL(2, \mathbb{C})$ , the maximal compact subgroup would become  $K = SO(3)$ , with  $K$ -types only of integral spin, but on the left, since  $G/K$  would no longer have a  $G$ -invariant spin structure,  $KO_{\bullet}(G/K)$  would be given by genuine representations of the double cover of  $K$ , i.e., representations only of half-integral spin.

**EXAMPLE 3.6.** The techniques we used in Example 3.3 and Theorem 3.5 can also be used to compute the reduced real  $C^*$ -algebras of arbitrary connected complex reductive Lie groups. A useful starting point is [50, Proposition 4.1], which states that for such a group  $G$ ,  $\widehat{G}_r$  is Hausdorff and  $C_r^*(G) \cong C_0(\widehat{G}_r) \otimes \mathcal{K}$  is a stable continuous-trace algebra with trivial Dixmier-Douady class. The cases of  $SO(4n, \mathbb{C})$ ,  $SO(2n+1, \mathbb{C})$ , and  $Sp(n, \mathbb{C})$  are particularly easy (and interesting). By [1, Theorem 7.7] or [10, VI.(5.4)(vi)], all representations of  $SO(2n+1)$  are of real type, and by [1, Theorem 7.9] or [10, VI.(5.5)(ix)], all representations of  $SO(4n)$  are of real type. Thus we obtain

**THEOREM 3.7.** *Let  $G = SO(2n+1, \mathbb{C})$  or  $SO(4n, \mathbb{C})$ . Then*

$$C_{r,\mathbb{R}}^*(G) \cong C_0^{\mathbb{R}}(\widehat{G}_r) \otimes \mathcal{K}_{\mathbb{R}}$$

*is a stable real continuous-trace algebra with trivial Dixmier-Douady class.*



PROOF. By Theorem 3.4, the involution on  $\widehat{G}_r$  is trivial, and the sign choice invariant has to be constant on each connected component. Let  $M$  be a Cartan subgroup of  $G$ , let  $W$  be its Weyl group, let  $B = MN$  be a Borel subgroup, let  $K$  be a maximal compact subgroup (an orthogonal group  $SO(2n+1)$  or  $SO(4n)$ ), and let  $T = K \cap M$ , a maximal torus in  $K$ . The irreducible representations in the reduced dual are all principal series  $\text{Ind}_B^G \chi$ , where  $\chi$  is a character of  $M$  extended to a character of  $B$  by taking it to be trivial on  $N$ . When restricted to  $K$ , this is the same as  $\text{Ind}_T^K \chi|_T$ . If  $\rho \in \widehat{K}$  is a  $K$ -type, then by Frobenius reciprocity,  $\rho$  appears in this induced representation with multiplicity equal to the multiplicity of  $\chi|_T$  in  $\rho|_T$ . So given  $\chi$ , take  $\rho$  to have highest weight in the  $W$ -orbit of  $\chi|_T$ , and we see that  $\rho$  occurs in  $\text{Ind}_B^G \chi$  with multiplicity 1. Since  $\rho$  is real and  $\text{Ind}_B^G \chi$  is of either real or quaternionic type, we see that its being of real type is the only possibility. (Otherwise the invariant skew-symmetric form on the representation would restrict to a skew-symmetric invariant form on  $\rho$ .) Thus  $C_{r,\mathbb{R}}^*(G)$  is a stable real continuous-trace algebra with all irreducible representations of real type. The Dixmier-Douady invariant has to vanish since as pointed out in [50, p. 277], all connected components of  $\widehat{G}_r$  are contractible.  $\square$

In a similar fashion we have

**THEOREM 3.8.** *Let  $G = Sp(n, \mathbb{C})$ , let  $M$  be a Cartan subgroup, and let  $W$  be its Weyl group. Let  $K = Sp(n)$ , a maximal compact subgroup, and let  $T = M \cap K$ , a maximal torus in  $K$ . Then  $C_{r,\mathbb{R}}^*(G)$  is a stable real continuous-trace algebra which is Morita equivalent to a direct sum of pieces of the form  $C_0^{\mathbb{R}}(Y)$  and  $C_0^{\mathbb{H}}(Y)$ . Here  $Y$  ranges over the components of  $\widehat{M}/W$ . Infinitely many pieces of each type (real or quaternionic) occur. If  $\chi \in \widehat{M}$  and  $\chi|_T$  is its “discrete parameter”, then the associated summand in  $C_{r,\mathbb{R}}^*(G)$  is of real type if and only if the representation of  $K$  with highest weight in the  $W$ -orbit of  $\chi$  is of real type.*

PROOF. This is exactly the same as the proof of Theorem 3.7, the only difference being that “half” of the representations of  $K$  are of quaternionic type (see [1, Theorem 7.6] and [10, VI.(5.3)(vi)]).  $\square$

**3.3. Orientifold string theories.** A last area where real  $C^*$ -algebras and their  $K$ -theory seem to play a significant role is in the study of orientifold string theories in physics. (See for example [51, §8.8] and [52, Ch. 13].) Such a theory is based on a spacetime manifold  $X$  equipped with an involution  $\iota$ , and the theory is based on a “sigma

model” where the fundamental “strings” are equivariant maps from a “string worldsheet”  $\Sigma$  (an oriented Riemann surface) to  $X$ , equivariant with respect to the “worldsheet parity operator”  $\Omega$ , an orientation-reversing involution on  $\Sigma$ , and  $\iota$ , the involution on  $X$ . Restricting attention only to equivariant strings is basically what physicists often call *GSO projection*, after Gliozzi-Scherk-Olive [23], and introduces enough flexibility in the theory to get rid of lots of unwanted states. In order to preserve a reasonable amount of supersymmetry, usually one assumes that the spacetime manifold (except for a flat Minkowski space factor, which we can ignore here) is chosen to be a Calabi-Yau manifold, that is, a complex Kähler manifold  $X$  with vanishing first Chern class, and that the involution of  $X$  is either holomorphic or anti-holomorphic. If we choose  $X$  to be compact, then in low dimensions there are very few possibilities — if  $X$  has complex dimension 1, then it is an elliptic curve, and if  $X$  has complex dimension 2, then it is either a complex torus or a  $K3$  surface. In the papers [14, 15], the case of an elliptic curve was treated in great detail. Example 2.9 and Theorem 2.10 were motivated by the case of  $K3$  surfaces, which should also be of great physical interest.

Now we need to explain the connection with real  $C^*$ -algebras and their  $K$ -theory. An orientifold string theory comes with two kinds of important submanifolds of the spacetime manifold  $X$ : *D-branes*, which are submanifolds on which “open” strings — really, compact strings with boundary — can begin or end (where we specify boundary conditions of Dirichlet or Neumann type), and *O-planes*, which are the connected components of the fixed set of the involution  $\iota$ . There are charges attached to these two kinds of submanifolds. D-branes have charges in  $K$ -theory [43, 72], where the kind of  $K$ -theory to be used depends on the specific details of the string theory, and should be a variant of  $KR$ -theory for orientifold theories. The O-planes have  $\pm$  signs which determine whether the Chan-Paton bundles restricted to them have real or symplectic type. These sign choices result in “twisting” of the  $KR$ -theory, such as appeared above in Definition 2.4. In addition, there is a further twisting of the  $KR$ -theory due to the “ $B$ -field” which appears in the Wess-Zumino term in the string action. It would be too complicated to explain the physics involved, but mathematically, the  $B$ -field gives rise to a class in Moutou’s  $H_t^2(X, \mathcal{T})$ . But in short, the effect of the O-plane charges and the  $B$ -field is to make the D-brane charges live in twisted  $KR$ -theory, i.e., in the  $K$ -theory of a real continuous-trace algebra determined by the O-plane charges and the  $B$ -field. In this way, (type II) orientifold string theories naturally

lead to  $K$ -theory of real continuous-trace algebras, which is most interesting from the point of view of physics when  $(X, \iota)$  is a Calabi-Yau manifold with a holomorphic or anti-holomorphic involution [14, 15].

An important aspect of string theories is the existence of *dualities* between one theory and another. These are cases where two seemingly different theories predict the same observable physics, or in other words, are equivalent descriptions of the same physical system. The most important examples of such dualities are *T-duality*, or target-space duality, where the target space  $X$  of the model is changed by replacing tori by their duals, and the very closely related *mirror symmetry* of Calabi-Yau manifolds. These dualities do not have to preserve the type of the theory (IIA or IIB) — in fact, in the case of T-duality in a single circle, the type is reversed — and they frequently change the geometry or topology of the spacetime and/or the twisting (sign choice and/or Dixmier-Douady class). The possible theories with  $X$  an elliptic curve and  $\iota$  holomorphic or anti-holomorphic were studied in [19, 14, 15], and found to be grouped into 3 classes, each containing 3 or 4 different theories. All of the theories in a single group are related to one another by dualities, and theories in two different groups can never be related by dualities. One way to see this is via the twisted  $KR$ -theory classifying the D-brane charges. Theories which are dual to one another must have twisted  $KR$ -groups which agree up to a degree shift, while if the  $KR$ -groups are non-isomorphic (even after a degree shift), the two theories cannot possibly be equivalent. Thus calculations of twisted  $KR$ -theory provide a methodology for testing conjectures about dualities in string theory.

In the case where  $X$  is an elliptic curve and  $\iota$  is holomorphic or anti-holomorphic, the twisted  $KR$ -groups were computed in [14, 15]. In one group of three theories ( $\iota$  the identity,  $\iota$  anti-holomorphic with a fixed set with two components and with trivial sign choice, and  $\iota$  holomorphic with four isolated fixed points), the  $KR$ -theory turned out to be  $KO^{-\bullet}(T^2)$ , up to a degree shift. In the next group ( $\iota$  holomorphic and free,  $\iota$  anti-holomorphic and free,  $\iota$  holomorphic with four fixed points with sign choice  $(+, +, -, -)$ , and  $\iota$  anti-holomorphic with a fixed set with two components and sign choice  $(+, -)$ ), the groups turned out to be  $KSC^{-\bullet} \oplus KSC^{-\bullet-1}$  up to a degree shift. In the last group ( $\iota$  the identity but the Dixmier-Douady invariant ( $B$ -field) nontrivial,  $\iota$  holomorphic with four isolated fixed points and sign choice  $(+, +, +, -)$ , and  $\iota$  anti-holomorphic with fixed set a circle), the  $KR$ -theory turned out to be  $KO^{-\bullet} \oplus KO^{-\bullet} \oplus K^{-\bullet-1}$  up to a degree shift. The  $KR$ -groups in one T-duality group are not isomorphic to those in another, so there cannot be any additional dualities between theories.

Rather curiously, it turns out (as was shown in [59]) that all of the isomorphisms of twisted  $KR$ -groups associated to dualities of elliptic curve orientifold theories arise from the real Baum-Connes isomorphisms for certain solvable groups with  $\mathbb{Z}$  or  $\mathbb{Z}^2$  as a subgroup of finite index. This suggests a rather mysterious connection between representation theory and duality for string theories, which we intend to explore further. It will be especially interesting to study dualities between orientifold theories compactified on abelian varieties of dimension 2 or 3 and on K3-surfaces, and ultimately on simply connected Calabi-Yau 3-folds.

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